IMPLICIT FUNCTION

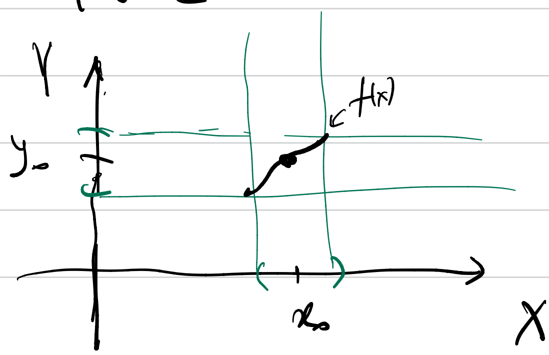
Thm X, Y, Z Banach, $F : \mathcal{U} \subseteq X \times Y \to Z$, $\mathcal{U}$ open
and assume
$$\exists\ (x_0, y_0) \in \mathcal{U} : \qquad F(x_0, y_0) = 0$$
Assume also
1) F is continuous in $\mathcal{U}$
2) F has y-partial derivative in $\mathcal{U}$ and
   $d_y F : \mathcal{U} \to \mathcal{L}(Y, Z)$ is continuous
3) $d_y F(x_0, y_0) \in \mathcal{L}(Y, Z)$ is invertible (with bd inverse)

Then
(i) $\exists\ r, \delta > 0$ and a unique continuous function
$$f : B_r^X(x_0) \longrightarrow B_\delta^Y(y_0)$$
with
$$f(x_0) = y_0 \qquad \text{and} \qquad F(x, f(x)) \equiv 0 \quad \forall x \in B_r^X(x_0)$$

Viceversa $\forall\ (x, y)$ sol in $B_r^X(x_0) \times B_\delta^Y(y_0)$ of $F(x,y) = 0$
we have $\quad y = f(x)$

(ii) If $F \in C^1(\mathcal{U}, Z) \Rightarrow f \in C^1$ in $\operatorname{int}(B_r^X(x_0))$
and
$$d_x f(x) = -\left[d_y F(x, f(x)) \right]^{-1} d_x F(x, f(x))$$

(iii) If $F \in C^k(\mathcal{U}, Z)$, $k > 1 \Rightarrow f \in C^k$



proof w.l.o.g. $(x_0, y_0) = (0,0)$. Write

$$F(x,y) = \underbrace{F(0,0)}_{\overset{\shortparallel}{0}} + \underbrace{d_y F(0,0)}_{A}[y] + \underbrace{F(x,y) - F(0,0) - d_y F(0,0)[y]}_{R(x,y)}$$

$$= A y + R(x,y)$$

So

$$F(x,y) = 0 \iff y = -\overset{\in \mathcal{L}(Z,Y)}{A^{-1}} R(x,y)$$

so given x sufficiently small, we want to find $\exists !\, y$ solving $y = -A^{-1} R(x,y)$

We show that the map

$$Y \longrightarrow Y$$
$$y \longrightarrow \Phi(x,y) := -A^{-1} R(x,y)$$

is a contraction in a suff. small ball in Y, for any x suff small

Lemma $\exists\, r, \delta > 0 : \forall x \in B^X_r(0)$, $\phi(x, \cdot)$ is a contraction in $B^Y_\delta(0)$

proof check that $\forall x \in B^X_r(0)$

(a) $\phi(x,\cdot) : B^Y_\delta(0) \longrightarrow B^Y_\delta(0)$

(b) $\| \phi(x,y_1) - \phi(x,y_2) \| \le \frac{1}{2} \|y_1 - y_2\| \quad \forall y_1, y_2 \in B^Y_\delta(0)$

Start with (b)

$$\| \phi(x,y_1) - \phi(x,y_2) \|_Y \le \| A^{-1} \|_{\mathcal{L}(Z,Y)} \| R(x,y_1) - R(x,y_2) \|_Z$$

$$R(x,y_1) - R(x,y_2) = F(x,y_1) - Ay_1 - F(x,y_2) + Ay_2$$

$$= F(x,y_1) - F(x,y_2) - A(y_1 - y_2)$$

$$= \int_0^1 d_y F(x, ty_1 + (1-t)y_2)\,[y_1 - y_2]\,dt \;-\; d_y F(0,0)\,[y_1 - y_2]$$

$$= \int_0^1 \Big(d_y F(x, ty_1 + (1-t)y_2) - d_y F(0,0) \Big) [y_1 - y_2]\, dt$$

By assumption, $d_y F$ is continuous as a map $\mathcal{U} \to \mathcal{L}(Y, Z)$

$\rightsquigarrow$ $\forall \varepsilon > 0$, $\exists\, r_\varepsilon, \delta_\varepsilon > 0$ st.

$$\forall x \in B^X_{r_\varepsilon}(0),\ \forall y_1, y_2 \in B^Y_{\delta_\varepsilon}(0) : \ \| d_y F(x, ty_1 + (1-t)y_2) - d_y F(0,0) \| \le \varepsilon$$

$$\rightsquigarrow \| R(x, y_1) - R(x, y_2) \| \le \varepsilon \, \| y_1 - y_2 \| \quad \forall\, x \in B^X_{r_\varepsilon}(0),\ y_1, y_2 \in B^Y_{\delta_\varepsilon}(0)$$

$$\rightsquigarrow \| \phi(x, y_1) - \phi(x, y_2) \| \le \|A^{-1}\|\, \varepsilon \, \| y_1 - y_2 \| \le \frac{1}{2} \| y_1 - y_2 \|$$

provided $\varepsilon = \dfrac{1}{2\|A^{-1}\|}$, $\forall x \in B^X_{r_\varepsilon}(0)$, $\forall y_1, y_2 \in B^Y_{\delta_\varepsilon}(0)$

This proves (b), now prove (a):

Let $y \in B^Y_{\delta_\varepsilon}(0)$:

$$\| \phi(x, y) \| \le \| \phi(x, 0) \| + \| \phi(x, y) - \phi(x, 0) \| \le \| \phi(x, 0) \| + \frac{1}{2} \| y \| \le \| \phi(x, 0) \| + \delta/2$$

Now write $\phi(x, 0) = -A^{-1} R(x, 0) = -A^{-1} F(x, 0) \overset{?}{\le} \dfrac{\delta}{2}$

Use that F is continuous at $(0,0)$ $\rightsquigarrow$ $\exists\, 0 < r < r_\varepsilon$ so that

$$\| F(x, 0) \| = \| F(x, 0) - \underbrace{F(0,0)}_{=0} \| \le \frac{\delta}{2\|A^{-1}\|}$$

provided $x \in B^X_r(0)$. $\Rightarrow \| \phi(x, y) \| \le \delta \quad \forall x \in B_r(0),\ \forall y \in B_\delta(0)$ $\square$

By Banach fixed point thm, we deduce
$\forall x \in B_2^X(0)$, $\exists!$ fixed point $y = f(x) \in B_\delta^Y(0)$ of the map $\phi(x,\cdot)$

i.e. $y = f(x)$ solves $f(x) = -A^{-1}R(x, f(x)) \iff F(x, f(x)) = 0$

Since $F(0,0) = 0$, we have $f(0) = 0$ by unicity of fixed point.

Moreover

If $(x,y) \in B_2^X(0) \times B_\delta^Y(0)$ sol of $F(x,y) = 0 \Rightarrow y \in B_\delta^Y(0)$ is a fixed point of $\phi(x,\cdot) \Rightarrow y = f(x)$ (unicity of fixed point)

<u>Continuity of $f(x)$</u>: $\forall x_1, x_2 \in B_2^X(0)$

$$\| f(x_1) - f(x_2) \| = \| \phi(x_1, f(x_1)) - \phi(x_2, f(x_2)) \|$$

$$\leq \| \phi(x_1, f(x_1)) - \phi(x_1, f(x_2)) \|$$

$$+ \| \phi(x_1, f(x_2)) - \phi(x_2, f(x_2)) \|$$

$$\leq \frac{1}{2} \| f(x_1) - f(x_2) \| + \| \phi(x_1, f(x_2)) - \phi(x_2, f(x_2)) \|$$

$$\Rightarrow \| f(x_1) - f(x_2) \| \leq 2 \|A^{-1}\| \, \| R(x_1, f(x_2)) - R(x_2, f(x_2)) \|$$

$$\leq 2 \|A^{-1}\| \, \| F(x_1, f(x_2)) - F(x_2, f(x_2)) \| \xrightarrow{x_1 \to x_2} 0 \text{ as } F \text{ continuous}$$

This proves item (i)

(ii) <u>Differentiability of $f(x)$</u> $\quad \Lambda := -[d_y F(x, f(x))]^{-1} d_x F(x, f(x))$

We need to prove $\frac{\| f(x+h) - f(x) - \lambda[h] \|}{\|h\|} \xrightarrow{\|h\|\to 0} 0$

Take $\|h\| \ll 1$ so that $x+h \in B_r^X(0)$

f continuous $\Rightarrow$ $k := f(x+h) - f(x) \to 0$ as $\|h\| \to 0$

F diff. at $(x, f(x))$ $\Rightarrow$ $\forall \varepsilon > 0, \exists \eta > 0 : \forall \|h\| + \|k\| < \eta$

$$\| \underbrace{F(x+h, y+k)}_{\substack{= F(x+h, f(x+h)) \\ = 0}} - \underbrace{F(x,y)}_{\substack{= F(x, f(x)) \\ = 0}} - dF(x,y)[h,k] \| \le \varepsilon (\|h\| + \|k\|)$$

at $y = f(x) \Rightarrow$

$$\Rightarrow \| d_x F(x,y)[h] + d_y F(x,y)[k] \| < \varepsilon (\|h\| + \|k\|)$$

Next, use that $d_y F(x, f(x)) \xrightarrow{x \to 0} d_y F(0,0)$

so since $d_y F(0,0)$ is invertible, so is $d_y F(x, f(x))$ provided $|x|$ suff small.

$$\rightsquigarrow \| [d_y F(x, f(x))]^{-1} \|_{\mathcal{L}(Z,Y)} \le 2 \| (d_y F(0,0))^{-1} \| \le 2 \|A^{-1}\|$$

↑ by Neumann series

So we can write

$$\underbrace{f(x+h) - f(x)}_{k} - \Lambda h = k + [d_y F(x, f(x))]^{-1} d_x F(x, f(x))[h]$$

$$\overset{y = f(x)}{=} [d_y F(x,y)]^{-1} \left(d_y F(x,y)[k] + d_x F(x,y)[h] \right)$$

$$\Rightarrow \| f(x+h) - f(x) - \Lambda h \| \le 2 \|A^{-1}\| \varepsilon \left(\|h\| + \| f(x+h) - f(x) \| \right)$$

$$\le 2 \|A^{-1}\| \varepsilon \left(\|h\| + \| f(x+h) - f(x) - \Lambda h \| + \|\Lambda h\| \right)$$

choose ε so that $2\|A^{-1}\|\varepsilon < 1/2$

$$\rightsquigarrow \quad \frac{1}{2}\|f(x+h)-f(x)-\Lambda h\| \leq 2\|A^{-1}\|\varepsilon\,\|h\|\,(1+\|\Lambda\|)$$

$$\rightsquigarrow \quad \frac{\|f(x+h)-f(x)-\Lambda h\|}{\|h\|} \leq C\varepsilon$$

Since ε is arbitrary, we deduce that it goes to 0

(viii) $F \in C^2 \Rightarrow d_x f(x) = \underbrace{[\underbrace{-d_y F(x, f(x))}_{\in C^1}]^{-1}}_{A \to A^{-1} \in C^1} \underbrace{d_x F(x, f(x))}_{\in C^1}$

$\rightsquigarrow d_x f \in C^1 \Rightarrow f \in C^2$

$\square$

LOCAL INVERSION MAPPING THEOREM

Thm $f: \mathcal{U} \subseteq X \to Y$, $f \in C^1$ and so that $\exists x_0 \in \mathcal{U}$: $df(x_0) \in \mathcal{L}(X, Y)$ is invertible with a bdd inverse

Then 1) f is locally invertible at x_0: $\exists\, \mathcal{U}_1 \ni x_0$, $\mathcal{V} \ni f(x_0) = y_0$ s.t. f is a diffeomorphism $f|_{\mathcal{U}_1}: \mathcal{U}_1 \to \mathcal{V}$

2) $f^{-1} \in C^1(\mathcal{V}, X)$, $df^{-1}(y_0) = [df(x_0)]^{-1}$

3) $f \in C^k \Rightarrow f^{-1} \in C^k$

proof $F: \mathcal{U} \times Y \to Y$, $F(x,y) = f(x) - y$

- $F(x_0, f(x_0)) = 0$
- $F \in C^1$
- $d_x F(x_0, f(x_0)) = d_x f(x_0)$ is invertible.

IFT $\Rightarrow$ $\exists\, r, \delta > 0$ and a map $g: B^Y_\delta(y_0) \to B^X_r(x_0)$

so that $F(g(y),y)=0 \quad \forall y \iff f(g(y))=y$

so $g=f^{-1}$. Other properties follows again by IFT.

$\square$

Semi-linear Sturm-Liouville problem

$$(D)\quad \begin{cases} -u''+f(u)=g & x\in[0,1] \\ u(0)=u(1)=0 \end{cases}$$

$f:\mathbb{R}\to\mathbb{R}$, $f(0)=0$, $f\in C^1$ (ex: $f(t)=t^{2n}$)

If g is small in norm $\rightarrow$ apply IFT

$$X=\{u\in C^2([0,1]):\ u(0)=u(1)=0\},\quad \|u\|_X=\sum_{j=0}^{2}\|\partial^j u\|$$

$$Y=\{u\in C^0([0,1]):\ \}\quad \|u\|_Y=\|u\|_{C^0}$$

$$F: X\times Y\to Y,\quad F(u,g)=-u''+f(u)-g$$

•) $F(0,0)=0$

•) regularity of F: $F(u,g)=\underset{-\frac{d^2}{dx^2}}{A}u+f(u)-g$

$u\to Au$ lin op $\leadsto$ enough $A\in\mathcal{L}(X,Y)$

$$\|Au\|_Y=\|u''\|_{C^0}\leq\|u\|_X$$

$C^2\to C^0$

$u\to f(u)$ Nemitski op: it is C^1 $\quad d f(u)[h]=f'(u)\circ h$

•) invertibility of $d_u F(0,0)$

$$d_u F(0,0)[h]=Ah+f'(0)h\quad\in\mathcal{L}(X,Y)$$

Given $g \in Y$, find $\exists!\ h \in X$ so that

$d_u F(0,0)[h] = g \iff \begin{cases} -h'' + f'(0)\,h = g \\ h(0) = h(1) = 0 \end{cases}$

From Sturm-Liouville + Fredholm

$\forall g \in L^2\ \ \exists!\ h \in H^1_0$ weak sol of $\begin{cases} -h'' + f'(0)\,h = g \\ u(0) = u(1) = 0 \end{cases}$ (D_L)

$\Updownarrow$

homogeneous problem has only the trivial sol

$\Updownarrow$

$f'(0) \notin \{ -k^2\pi^2,\ k \in \mathbb{N} \}$ $(\ast)$

So assume $(\ast)$

$\Rightarrow\ \forall g \in Y,\ \exists!\ h$ weak sol of (D_L), $h \in H^1_0$

$\rightsquigarrow h \in C^0 \rightsquigarrow h'' = \underbrace{f'(0)\,h - g}_{\in C^0} \Rightarrow h \in C^2$

$\rightsquigarrow h$ classical sol & $h \in X$

$\rightsquigarrow d_u F(0,0)$ is bijective $\Rightarrow [d_u F(0,0)]^{-1} \in \mathcal{L}(Y,X)$

continuous $X \to Y$

Apply IFT $\rightsquigarrow$ solvability for $\|g\| \ll 1$.

Any datum g: continuity method

(H1) $f: \mathbb{R} \to \mathbb{R}$, $f \in C^1$, $f(0) = 0$, $f'(t) \geq 0$ $\forall t \in \mathbb{R}$

(in particular $f'(0) \notin \{-k^2\pi^2\}_{k \in \mathbb{N}}$)

Prop Assume (H1), then $\forall g \in C^0$, $\exists!\, u \in C^2$ sol of

$$(D) \quad \begin{cases} -u'' + f(u) = g \\ u(0) = u(1) = 0 \end{cases}$$

proof 1. <u>uniqueness</u>: Suppose given g, $\exists\, u, v \in C^2$ sol of (D)

$$\rightsquigarrow \quad -u'' + f(u) = -v'' + f(v)$$

$$\rightsquigarrow \quad -(u-v)'' + f(u) - f(v) = 0$$

Call $w = u - v$, then

$$f(u) - f(v) = \underbrace{\int_0^1 f'(s\,u(x) + (1-s)\,v(x))\,ds}_{a(x)} \; (u(x) - v(x))$$

$$\rightsquigarrow \quad \begin{cases} -w'' + a(x)\, w = 0 \\ w(0) = w(1) = 0 \end{cases}, \quad a(x) \geq 0$$

Conclude $w = 0$: multiply by w and $\int$:

$$0 = \int (w')^2 + \int \underbrace{a(x)}_{\geq 0} w^2(x)\,dx \geq \int (w')^2 \Rightarrow \begin{cases} w = \text{const} \\ w(0) = w(1) = 0 \end{cases}$$

$$\Rightarrow \quad w = 0$$

2. <u>Existence</u>: Call $G : X \to Y$, $G(u) = -u'' + f(u)$

we know G is C^1

goal: Im G open
Im G closed $\implies$ Im $G = Y$
Im G not empty

•) Im G not empty: trivial

•) Im G open: take $g_0 \in \text{Im } G \rightsquigarrow \exists u_0 \in X : G(u_0) = g_0$

$$dG(u_0)[h] = -h'' + f'(u_0) \cdot h$$

If $dG(u_0)$ is invertible $\rightsquigarrow$ apply inverse function theorem and obtain that Im G open:

Is it invertible? $\forall g \in Y$, find $1!$ $h \in X$ with $dG(u_0)[h] = g$

$$\iff \begin{cases} -h'' + f'(u_0) \cdot h = g \\ h(0) = h(1) = 0 \end{cases}$$

It's Sturm-Liouville problem $V(x) = f'(u_0(x)) \geq 0$, and we proved $\exists !$ sol $h \in C^2$

$\rightsquigarrow$ apply inverse function thm $\rightsquigarrow$ G is locally invertible around g_0 $\rightsquigarrow$ $\exists\, \mathcal{U}_{u_0}, \mathcal{V}_{g_0}$: $G: \mathcal{U}_{u_0} \to \mathcal{V}_{g_0}$ is bijective $\rightsquigarrow$ $\mathcal{V}_{g_0} \subseteq \text{Im } G$

•) Im G closed: take $(g_n)_n \in \text{Im } G$, $g_n \xrightarrow{Y} g \overset{?}{\Rightarrow} g \in \text{Im } G$

Take a seq $(u_n)_n \subseteq H_0^1$: $G(u_n) = g_n$. We need to extract from $(u_n)_n$ a converg. subseq and show that the limit solves the weak problem

Take $u_n \in X$ with $G(u_n) = g_n$ $\iff \begin{cases} -u_n'' + f(u_n) = g_n \\ u_n(0) = u_n(1) = 0 \end{cases}$

$-u_n'' + f(u_n)$

multiply by u_n and $\int$:

$$\int (u_n')^2 + \int f(u_n)\cdot u_n = \int g_n u_n$$

$$\rightsquigarrow \int (u_n')^2 + \int \underbrace{(f(u_n) - f(0))u_n}_{\geq 0} = \int (g_n - f(0))u_n$$

$$\rightsquigarrow \int (u_n')^2 \leq \int (g_n - f(0))\, u_n \leq \left(\int |g_n - f(0)|^2\right)^{1/2} \left(\int u_n^2\right)^{1/2}$$

$(\ast)$ $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$

$$\leq \varepsilon \int u_n^2 + \frac{1}{4\varepsilon} \int |g_n - f(0)|^2$$

Poincaré ineq:

$$\leq C\varepsilon \int (u_n')^2 + \frac{1}{4\varepsilon} \int |g_n - f(0)|^2$$

$$\rightsquigarrow (1 - C\varepsilon) \int (u_n')^2 \leq \frac{1}{4\varepsilon} \int |g_n - f(0)|^2 \leq C_1$$

$\rightsquigarrow \|u_n\|_{H_0^1} \leq C_1 \Rightarrow$ precompactness in C^0

$u_{n_k} \longrightarrow u$ in C^0

$u_{n_k} \rightharpoonup u$ in H_0^1

$$\int u_{n_k}' \varphi' + \int f(u_{n_k})\varphi = \int g_{n_k}\varphi \quad \forall \varphi \in H_0^1$$

$f(u_{n_k}) \to f(u)$ in C^0

$$\int u' \varphi' \qquad \int f(u)\varphi \qquad \int g\varphi \qquad \forall \varphi \in H_0^1$$

$\rightsquigarrow$ u solves the weak problem with data g

$u' \in H^1$ & $u'' = f(u) - g \in C^0 \Rightarrow u \in C^2$

$u(0) = u(1) = 0$ (from the C^0 convergence)

$\rightsquigarrow$ u strong sol and $G(u) = g$ $\Leftarrow$ Im G